Double-Angle Identities

$$\sin 2u = 2\sin u\cos u$$

$$\cos 2u = \begin{cases}\cos^2 u - \sin^2 u \\ 2\cos^2 u - 1 \\ 1 - 2\sin^2 u\end{cases}$$

$$\tan 2u = \frac{2\tan u}{1-\tan^2 u}$$

Power-Reducing Identities

$$\sin^2 u = \frac{1-\cos 2u}{2}$$

$$\cos^2 u = \frac{1+\cos 2u}{2}$$

$$\tan^2 u = \frac{1-\cos 2u}{1+\cos 2u}$$

Half-Angle Identities

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}}$$

$$\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$$

$$\tan\frac{u}{2} = \begin{cases}\pm\sqrt{\dfrac{1-\cos u}{1+\cos u}} \\ \dfrac{1-\cos u}{\sin u} \\ \dfrac{\sin u}{1+\cos u}\end{cases}$$

Example 1 **Prove each identify using a sum or difference formula.**

A. $\sin 2u = 2\sin u\cos u$

$\sin(u+u) = \sin u\cos u + \cos u\sin u$

$= 2\sin u\cos u$

B. $\cos^2\theta - \sin^2\theta = \cos 2\theta$

$\cos(\theta+\theta) = \cos\theta\cdot\cos\theta - \sin\theta\cdot\sin\theta$

$= \cos^2\theta - \sin^2\theta$

$\cos^2\theta - (1-\cos^2\theta)$ $\quad 1-\sin^2\theta - \sin^2\theta$

$\cos^2\theta - 1 + \cos^2\theta$ $\quad 1 - 2\sin^2\theta$

$2\cos^2\theta - 1$

Example 2 **Solve each equation over $[0, 2\pi)$.**

A. $\sin^2 x = 2\sin^2\left(\frac{x}{2}\right)$

$\sin^2 x = \cancel{2}\left(\frac{1-\cos 2\cdot\frac{x}{\cancel{2}}}{\cancel{2}}\right)$

$\sin^2 x = 1-\cos x$

$1-\cos^2 x = 1-\cos x$

$0 = \cos^2 x - \cos x$

$\cos x(\cos x - 1) = 0$

$\cos x = 0 \quad \cos x - 1 = 0$

$x = \cos^{-1}(0) \quad x = \cos^{-1}(1)$

$\boxed{x = \frac{\pi}{2}, \frac{3\pi}{2}} \quad \boxed{x = 0}$

B. $\sin 2x = \cos x$

$2\sin x\cos x = \cos x$

$2\sin x\cos x - \cos x = 0$

$\cos x(2\sin x - 1) = 0$

$\cos x = 0 \quad 2\sin x - 1 = 0$

$x = \cos^{-1}(0) \quad x = \sin^{-1}\left(\frac{1}{2}\right)$

$\boxed{x = \pi/2, 3\pi/2} \quad \boxed{x = \frac{\pi}{6}, \frac{5\pi}{6}}$

Example 3 **Use half-angle identities to find the exact value of each function.**

A. $\tan 15° = \tan\left(\frac{30°}{2}\right) = \frac{1-\cos 30°}{\sin 30°} = \frac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \left(1-\frac{\sqrt{3}}{2}\right)\cdot 2 = 2-\sqrt{}$

S	A
T	C

B. $\sin 75°$ = 1st quad

$= \sin\left(\frac{150°}{2}\right) = +\sqrt{\frac{1-\cos 150°}{2}} = \sqrt{\frac{1-\frac{-\sqrt{3}}{2}}{2}} = \sqrt{\frac{\frac{2+\sqrt{3}}{2}}{2}}$

C. $\cos\left(\frac{\pi}{8}\right) = \cos\left(\frac{\frac{\pi}{4}}{2}\right) = +\sqrt{\frac{1+\cos\pi/4}{2}} = \sqrt{\frac{2+\sqrt{3}}{2}\cdot\frac{1}{2}} = \sqrt{\frac{2+\sqrt{3}}{4}} = \sqrt{}$

$= \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{\frac{2+\sqrt{2}}{2}}{2}} = \sqrt{\frac{2+\sqrt{2}}{2}\cdot\frac{1}{2}} = \sqrt{\frac{2+\sqrt{2}}{4}} = \boxed{\frac{\sqrt{2+\sqrt{2}}}{2}}$

D. $\cos\frac{11\pi}{12} = \cos(165°) = \cos\left(\frac{330°}{2}\right) = -\sqrt{\frac{1+\cos 330°}{2}} = -\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}$

2nd quad

$-\sqrt{\frac{\frac{2+\sqrt{3}}{2}}{2}} = -\sqrt{\frac{2+\sqrt{3}}{2}\cdot\frac{1}{2}} = \boxed{-\frac{\sqrt{2+\sqrt{3}}}{2}}$

Example 4 Use double angle identities to find the exact value of each.

A. $\tan 60° = \tan(2\cdot 30°) = \dfrac{2\tan 30°}{1-\tan^2 30°} = \dfrac{2\sqrt{3}/3}{1-\left(\frac{\sqrt{3}}{3}\right)^2} = \dfrac{\frac{2\sqrt{3}}{3}}{1-\frac{3}{9}} = \dfrac{\frac{2\sqrt{3}}{3}}{\frac{2}{3}}$

$\dfrac{2\sqrt{3}}{3}\cdot\dfrac{3}{2} = \boxed{\sqrt{3}}$

B. $\sin 120° = \sin(2\cdot 60°) = 2\sin 60°\cdot\cos 60°$

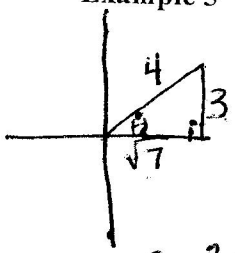
$= 2\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2} = \boxed{\frac{\sqrt{3}}{2}}$

C. $\cos\left(\frac{4\pi}{3}\right) = \cos\left(2\cdot\frac{2\pi}{3}\right) = \cos^2\left(\frac{2\pi}{3}\right) - \sin^2\left(\frac{2\pi}{3}\right)$

$\left(-\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2$

$\frac{1}{4} - \frac{3}{4} = -\frac{2}{4} = \boxed{-\frac{1}{2}}$

Example 5 If $\sin\theta = \frac{3}{4}$ and θ has its terminal side in the first quadrant, find the exact value of each function.



$m^2+3^2=4^2$

$m^2=7$

$m=\sqrt{7}$

a. $\sin 2\theta$

$2\sin\theta\cos\theta$

$\cancel{2}\left(\frac{3}{\cancel{4}_2}\right)\left(\frac{\sqrt{7}}{4}\right)$

$\boxed{\frac{3\sqrt{7}}{8}}$

b. $\cos 2\theta$

$1-2\sin^2\theta$

$1-2\left(\frac{3}{4}\right)^2$

$1-\cancel{2}^1\cdot\frac{9}{\cancel{16}_8}$

$1-\frac{9}{8}$

$\boxed{-\frac{1}{8}}$

c. $\tan 2\theta$

$\dfrac{2\tan\theta}{1-\tan^2\theta}$

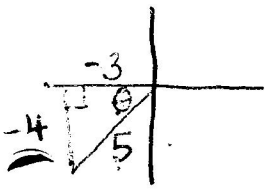
$\dfrac{2\left(\frac{3}{\sqrt{7}}\right)}{1-\left(\frac{3}{\sqrt{7}}\right)^2}$

$\dfrac{\frac{6}{\sqrt{7}}}{1-\frac{9}{7}} = \dfrac{\frac{6}{\sqrt{7}}}{-\frac{2}{7}} = \frac{\cancel{6}^3}{\sqrt{7}}\cdot\frac{-7}{\cancel{2}} = \boxed{\frac{-21}{\sqrt{7}}}$

Example 6 If $\tan\theta = \frac{4}{3}$ and $\pi \le \theta \le \frac{3\pi}{2}$, find the exact value of each function.

$\frac{\pi}{2} \le \frac{\theta}{2} \le \frac{3\pi}{4}$

2nd quad



a. $\sin\left(\frac{\theta}{2}\right)$

$+\sqrt{\dfrac{1-\cos\theta}{2}}$

$\sqrt{\dfrac{1-\frac{-3}{5}}{2}}$

$\sqrt{\dfrac{\frac{8}{5}}{2}}$

$\sqrt{\frac{\cancel{8}^4}{5}\cdot\frac{1}{\cancel{2}}} = \sqrt{\frac{4}{5}} = \boxed{\frac{2}{\sqrt{5}}}$

b. $\cos\left(\frac{\theta}{2}\right)$

$-\sqrt{\dfrac{1+\cos\theta}{2}}$

$-\sqrt{\dfrac{1-\frac{3}{5}}{2}}$

$-\sqrt{\dfrac{\frac{2}{5}}{2}}$

$-\sqrt{\frac{2}{5}\cdot\frac{1}{2}}$

$-\sqrt{\frac{1}{5}} = \boxed{-\frac{1}{\sqrt{5}}}$

c. $\tan\left(\frac{\theta}{2}\right)$

$\dfrac{1-\cos\theta}{\sin\theta}$

$\dfrac{1-\left(-\frac{3}{5}\right)}{-\frac{4}{5}}$

$\dfrac{\frac{8}{5}}{-\frac{4}{5}}$

$\frac{8}{\cancel{5}}\cdot\frac{-\cancel{5}}{4} = \boxed{-2}$