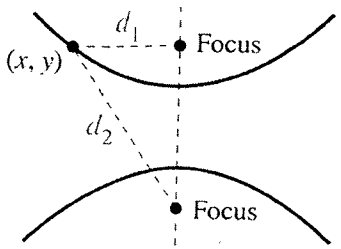
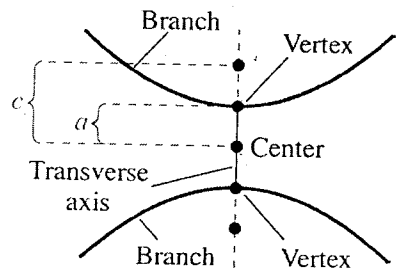
Hyperbolas

Definition of a Hyperbola

A **hyperbola** is the set of all points (x, y) in a plane, the difference of whose distances from two distinct fixed points, the **foci,** is a positive constant.



$d_2 - d_1$ is a positive constant.



- The graph of a hyperbola has two disconnected parts (**branches**)
- The line through the two foci intersects the hyperbolas at two points (**vertices**)
- The line segment connecting the vertices is the **transverse axis**, and the midpoint of the transverse axis is the **center** of the hyperbola

The General Form of a Hyperbola

- The general second-degree equation for *all* conics is $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.
- For ~~ellipses~~ hyperbolas, $B = 0$; A and C have different signs
  - $Ax^2 + Cy^2 + Dx + Ey + F = 0$

Standard Equation of a Hyperbola

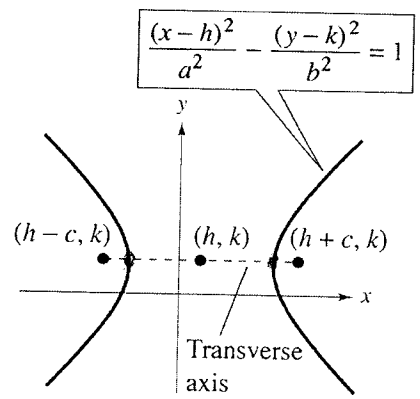
The **standard form of the equation of a hyperbola** with center at (h, k) is

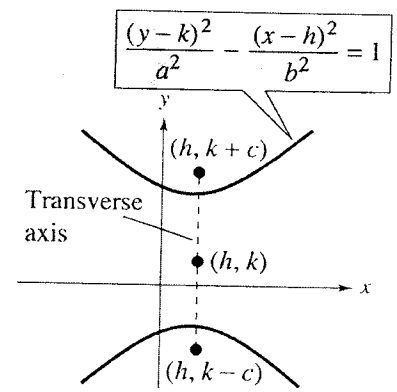
$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \quad \text{Transverse axis is horizontal.}$$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1. \quad \text{Transverse axis is vertical.}$$

The vertices are a units from the center, and the foci are c units from the center. Moreover, $c^2 = a^2 + b^2$. If the center of the hyperbola is at the origin $(0, 0)$, the equation takes one of the following forms.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{Transverse axis is horizontal.} \qquad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \quad \text{Transverse axis is vertical.}$$





Example 1

Convert $-16x^2 + y^2 - 32x - 4y - 28 = 0$ into standard form.

$$-16x^2 - 32x \quad + y^2 - 4y \quad = 28$$

$$-16(x^2 + 2x + 1) + y^2 - 4y + 4 = 28 - 16 + 4$$

$$\frac{-16(x+1)^2 + (y-2)^2}{16} = \frac{16}{16}$$

$$\frac{(y-2)^2}{16} - \frac{(x+1)^2}{1} = 1 \qquad \text{center } (-1, 2)$$

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Asymptotes of a Hyperbola

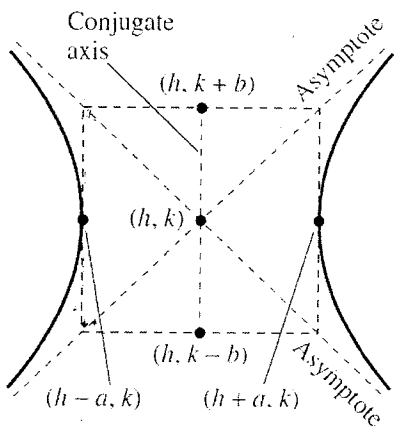
- Each hyperbola has two **asymptotes** that intersect at the center of the hyperbola
- The asymptotes pass through the corners of a rectangle of dimensions $2a$ by $2b$, with its center at (h, k)
- The **conjugate axis** of a hyperbola is the line segment of length $2b$ joining $(h, k+b)$ and $(h, k-b)$ if the transverse axis is horizontal
- The **conjugate axis** of a hyperbola is the line segment of length $\cancel{2b}$ joining $(h+b, k)$ and $(h-b, k)$ if the transverse axis is vertical

Asymptotes of a Hyperbola

$y = k \pm \frac{b}{a}(x-h)$ Asymptotes for horizontal transverse axis

$y = k \pm \frac{a}{b}(x-h)$ Asymptotes for vertical transverse axis

- The slopes for the asymptotes are always $\pm\frac{\text{rise}}{\text{run}}$
- The "run" always corresponds with the denominator of the x-term in the standard equation
  - If the transverse axis is horizontal, the hyperbola opens to the left/right, and the slopes are $\pm\frac{b}{a}$ because $\frac{(x-h)^2}{a^2}$ in the standard equation
  - If the transverse axis is vertical, the hyperbola opens up/down, and the slopes are $\pm\frac{a}{b}$ because $\frac{(x-h)^2}{b^2}$ in the standard equation



Graphing a Hyperbola

- Convert the equation into standard form.
- Plot the center (h, k).
- If the transverse axis is parallel to the x-axis:
  - Plot the vertices $(h \pm a, k)$.
- If the transverse axis is parallel to the y-axis:
  - Plot the vertices $(h, k \pm a)$.
- Determine the equations of the asymptotes.
- Plot the asymptotes.
- Draw the hyperbola.

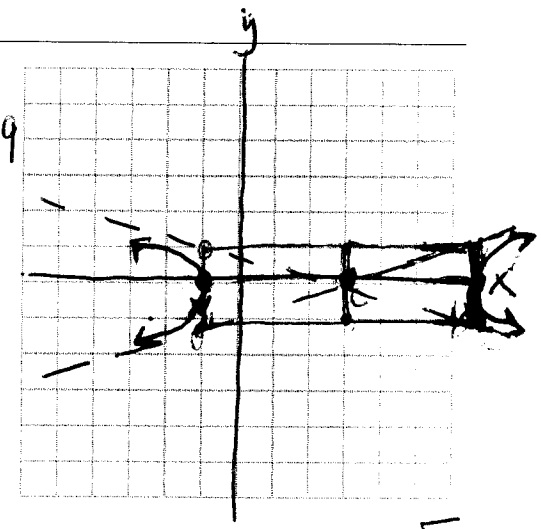
Example 2

Graph $x^2 - 16y^2 - 6x - 7 = 0$.

$x^2 - 6x + 9 - 16y^2 = 7 + 9$

$\frac{(x-3)^2 - 16y^2}{16} = \frac{16}{16}$

$\frac{(x-3)^2}{16} - \frac{y^2}{1} = 1$



$c(3,0)$

vertices $(-1,0)(7,0)$

$c^2 = a^2 + b^2$

$c^2 = 16 + 1$

$c^2 = 17$

$c = \sqrt{17}$

$ecc = \frac{c}{a} = \frac{\sqrt{17}}{4} = 1.03$

asymptotes:

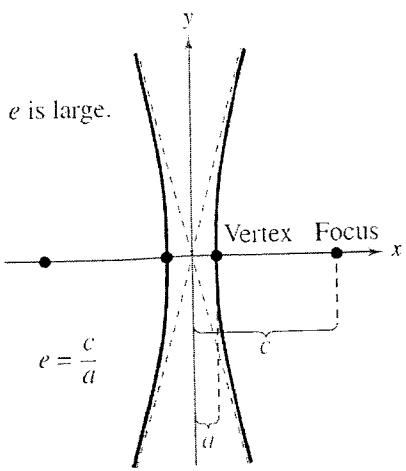
$y = 0 \pm \frac{1}{4}(x-3)$

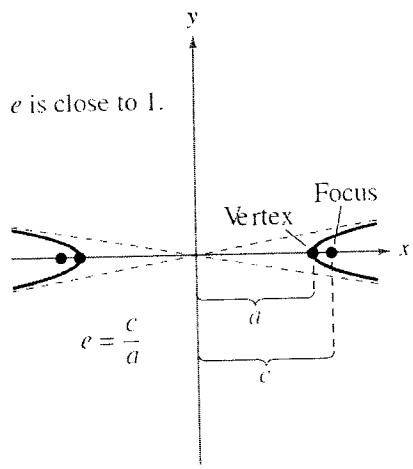
foci: $(3+\sqrt{17}, 0)$

$(3-\sqrt{17}, 0)$

More on Hyperbolas

- The **foci** are located along the transverse axis. It follows that $0 < a < c$.
- Note that $c^2 = a^2 + b^2$. (This is <u>not</u> the same as with the ellipse.)
- As with ellipses, the **eccentricity** of a hyperbola is $e = \frac{c}{a}$.
- For hyperbolas, $e > 1$.
- If the eccentricity is large, the branches of the hyperbola are nearly flat.
- If the eccentricity is close to 1, the branches of the hyperbola are more pointed.





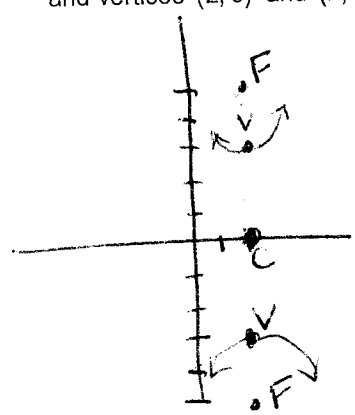
Example 3

What is the eccentricity of the hyperbola from Example 2?

1.03

Example 4

Write the equation in standard form of a hyperbola with foci $(2, 5)$ and $(2, -5)$, and vertices $(2, 3)$ and $(2, -3)$.



$C(2,0)$

$$\boxed{\frac{(y-0)^2}{9} - \frac{(x-2)^2}{16} = 1}$$

$c^2 = a^2 + b^2$

$25 = 9 + b^2$

$16 = b^2$

Classification of Conics

- Using the general form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$:

Conic	Test	Notes
Circle	$A = C$	$A \neq 0$
Parabola	$AC = 0$	$A = 0$ or $C = 0$, but not both
Ellipse	$AC > 0$	A and C have the same signs
Hyperbola	$AC < 0$	A and C have different signs

Example 5

Classify each of the following without converting to standard form:

a) $4x^2 + 5y^2 - 9x + 8y = 0$ ellipse

b) $2x^2 - 5x + 7y - 8 = 0$ parabola

c) $7x^2 + 7y^2 - 9x + 8y - 16 = 0$ circle

d) $4x^2 - 5y^2 - x + 8y + 1 = 0$ hyperbola

Summarizing Eccentricity

Interval	Curve	c	e	Note
$e = 0$	circle	0	0	
$0 < e < 1$	ellipse	$\sqrt{a^2 - b^2}$	c/a	Closer to 0—more circular Closer to 1—more elongated
$e = 1$	parabola	p	1	
$e > 1$	hyperbola	$\sqrt{a^2 + b^2}$	c/a	Closer to 1—more pointed Larger values—more flat