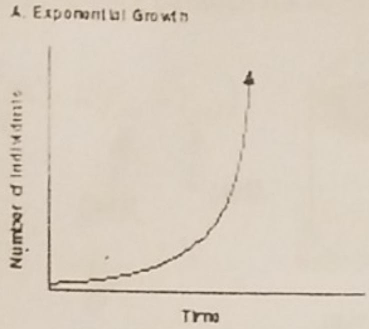
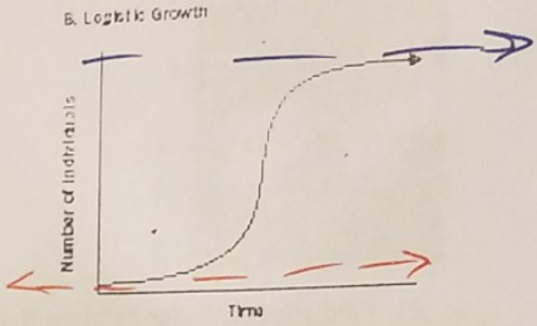
Notes---- Logistic Differential Equations

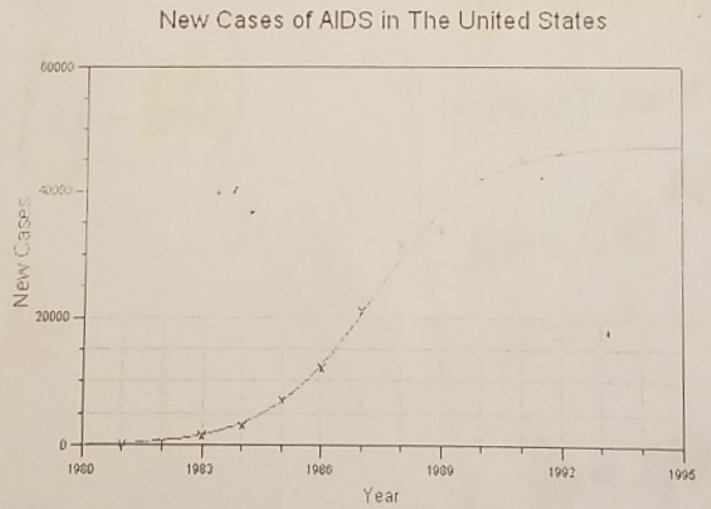
The exponential is only bounded below. However, for population growth there exists some upper limit past which growth cannot occur.



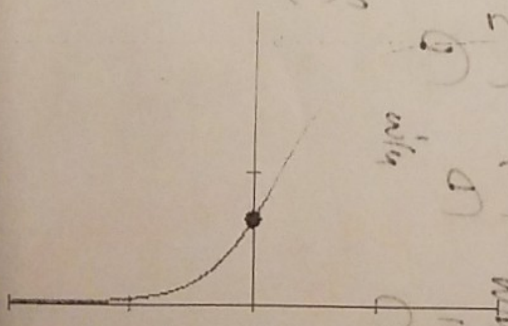


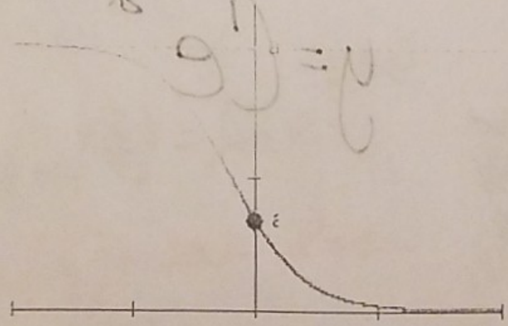
A logistic differential equation has the form: $\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$

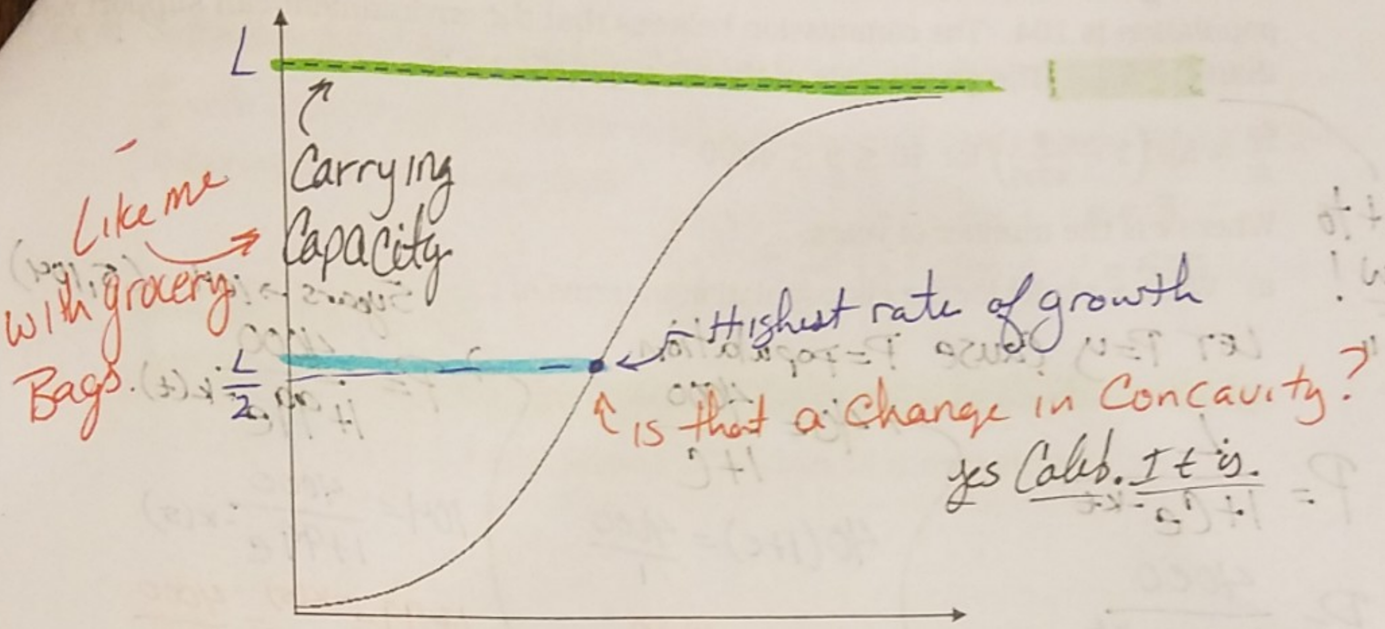
Where k and L are positive constants. L is the Carrying Capacity or the Limit to Grow which can be sustained or supported as time t increases.



Note: If y is between 0 and the carrying capacity L, then $\frac{dy}{dt} > 0$ and the population increases. If $y > L$ the $\frac{dy}{dt} < 0$ and the population decreases.







Finding the solution of the logistic equation.

Ex1) $\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$

$$\int \frac{dy}{y(1-\frac{y}{L})} = \int k\,dt$$

wait! What?

Look → $\frac{dy}{dt} = \frac{ky(1-\frac{y}{L})}{1}$

$$\frac{dt\,ky(1-\frac{y}{L})}{y(1-\frac{y}{L})} = \frac{dy\,(1)}{y(1-\frac{y}{L})}$$

$$\boxed{k\,dt = \frac{dy}{y(1-\frac{y}{L})}}$$

$$-\ln|L-y| + \ln|y| = kt + C$$

or

$$\ln|L-y| - \ln|y| = -kt + C$$

$$e^{\ln\left|\frac{L-y}{y}\right|} = e^{-kt+C}$$

Whoa! again

$$\ln|L-y| - \ln|y|$$

minus means (÷) $\ln\left|\frac{L-y}{y}\right|$

$$\frac{L-y}{y} = Ce^{-kt}$$

$$\frac{L}{y} - \frac{y}{y} = Ce^{-kt}$$

$$\frac{L}{y} - 1 = Ce^{-kt}$$

$$\frac{L}{y} = 1 + Ce^{-kt}$$

$$y = \frac{L}{1+Ce^{-kt}}$$

Ex 2) A state game commission releases 40 elk into a game refuge. After 5 years, the population is 104. The commission believes that the environment can support no more than 4000 elk. The growth rate of the elk population p is

$$\frac{dp}{dt} = kp\left(1 - \frac{p}{4000}\right) \text{ for } 40 \le p \le 4000$$

Limit to grow! "L"

Where t is the number of years.

a) Write a model for the elk population in terms of t.

5 years → 104 (5,104)

LET P=y Cause P=population

$$P = \frac{L}{1+Ce^{-kt}}$$

$$P = \frac{4000}{1+Ce^{-kt}}$$

$$40 = \frac{4000}{1+Ce^{-k(0)}}$$

$$40 = \frac{4000}{1+C}$$

$$40(1+C) = \frac{4000}{1}$$

$$1+C = \frac{4000}{40}$$

$$C = 99$$

$$P = \frac{4000}{1+99e^{-k(t)}}$$

$$104 = \frac{4000}{1+99e^{-k(5)}}$$

$$1+99e^{-k(5)} = \frac{4000}{104}$$

$$e^{-k(5)} = \frac{\frac{4000}{104}-1}{99} \Rightarrow k = 0.19436102$$

$$P = \frac{4000}{1+99e^{-0.194(t)}}$$

b) Use the model to estimate the elk population after 15 years.

$$P(15) = \frac{4000}{1+99e^{-0.194(15)}} \Rightarrow P(15) = 628.538 \text{ elks}$$

c) Find the limit of the model as $t \to \infty$.

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{4000}{1+99e^{-0.194(t)}} = 4000 \text{ elks}$$

Ex 3) Suppose the population of the bears in a national park grows according to the logistic differential equation. $\frac{dP}{dt} = 5P - 0.002P^2$, where P is the number of bears in time t years.

a) If $P(0) = 100$, then the $\lim_{t\to\infty} P(t) =$ 2500

b) If $P(0) = 1500$, then the $\lim_{t\to\infty} P(t) =$ 2500

c) If $P(0) = 3000$, then the $\lim_{t\to\infty} P(t) =$ 2500

$$\frac{dP}{dt} = 5P\left(1 - \frac{P}{2500}\right) \text{ GCF}$$

$k = 5$

$L = 2500$

d) How many bears are in the park when the population is growing the fastest?

$$\frac{1}{2}(2500) = 1250$$

Ex 4) Suppose a population of wolves grows to the logistic differential equation $\frac{dP}{dt} = 3P - 0.01P^2$ where P is the number of wolves at time t years. Which of the following statements are true?

$$\frac{dP}{dt} = 3P\left(1 - \frac{P}{300}\right) \quad k = 3 \quad L = 300$$

I. $\lim_{t \to \infty} P(t) = 300$

II. The growth rate of the wolf population is greatest at $P = 150$.

III. If $P > 300$, the population of wolves is increasing

a) I only
b) II only
c) I and II
d) ~~II and III~~
e) All of them.

Ex 5) A population of animals growth is modeled by a function P that satisfies the logistic differential equation $\frac{dP}{dt} = 0.01P(100 - P)$, where time t is measured in years.

a) If $P(0) = 20$, solve for P as a function of t.
b) Using the answer in part a), find P when $t = 3$ years.
c) Using the answer in part a), find t when $P = 80$ animals.

$k = 1 \quad L = 100$

$$\frac{dP}{dt} = 0.01P(100 - P)$$

$$\frac{dP}{dt} = P[0.01](100 - P)$$

$$\frac{dP}{dt} = P\left(1 - \frac{P}{100}\right)$$

$$P(t) = \frac{L}{1 + Ce^{-kt}}$$

$$P(t) = \frac{100}{1 + Ce^{-(1)t}}$$

a) $20 = \frac{100}{1 + Ce^{-0}}$

$20 = \frac{100}{1 + C}$

$1 + C = \frac{100}{20}$

$\boxed{C = 5 - 1 = 4}$

$P(t) = \frac{100}{1 + 4e^{-t}}$

b) $P(3) = \frac{100}{1 + 4e^{-3}}$

$= 83.393$

≈ 83 ANIMALS

c) $80 = \frac{100}{1 + 4e^{-t}}$

$t = 2.773$ años