

Sum and Difference Identities

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

$$\sin(a-b) = \sin a \cos b - \cos a \sin b$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$

$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$$

Example 1 Use sum and difference formulas to simplify each expression.

A. $\sin 22^\circ \cos 13^\circ + \cos 22^\circ \sin 13^\circ$

$$\sin(22^\circ + 13^\circ)$$

$$\sin 35^\circ$$

B. $\cos(\pi/3) \cos(\pi/4) - \sin(\pi/3) \sin(\pi/4)$

$$\cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$$

$$\cos\left(\frac{4\pi+3\pi}{12}\right)$$

$$\cos\left(\frac{7\pi}{12}\right)$$

B. $\sin(\pi/2 - x) = \cos x$

$$\begin{aligned} &\cos\frac{\pi}{2} \cdot \cos x + \sin\frac{\pi}{2} \cdot \sin x \\ &0 \cdot \cos x + 1 \cdot \sin x \\ &\sin x \end{aligned}$$

$$\begin{aligned} &\sin\frac{\pi}{2} \cdot \cos x - \cos\frac{\pi}{2} \cdot \sin x \\ &1 \cdot \cos x - 0 \cdot \sin x \\ &\cos x \end{aligned}$$

C. $\sin(x + \pi) = -\sin x$

$$\begin{aligned} &\sin x \cdot \cos \pi + \cos x \cdot \sin \pi \\ &\sin x \cdot -1 + \cos x \cdot 0 \\ &-\sin x \end{aligned}$$

D. $\cos\left(x + \frac{3\pi}{2}\right) = \sin x$

$$\begin{aligned} &\cos x \cdot \cos \frac{3\pi}{2} - \sin x \cdot \sin \frac{3\pi}{2} \\ &\cos x \cdot 0 - \sin x \cdot -1 \\ &\sin x \end{aligned}$$

Example 3 Find the EXACT value of each expression.

A. $\cos 75^\circ = \cos(30^\circ + 45^\circ) = \cos 30^\circ \cos 45^\circ - \sin 30^\circ \sin 45^\circ$

$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \boxed{\frac{\sqrt{6}-\sqrt{2}}{4}}$$

B. $\sin 75^\circ = \sin(30^\circ + 45^\circ) = \sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ$

$$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \boxed{\frac{\sqrt{2}+\sqrt{6}}{4}}$$

C. $\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\frac{\pi}{3} \cos\frac{\pi}{4} + \sin\frac{\pi}{3} \sin\frac{\pi}{4}$

$$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \boxed{\frac{\sqrt{2}+\sqrt{6}}{4}}$$

$$D. \tan 255^\circ = \tan(210^\circ + 45^\circ) = \frac{\tan 210^\circ + \tan 45^\circ}{1 - \tan 210^\circ \tan 45^\circ} = \frac{\frac{1}{\sqrt{3}} + \frac{-1}{2}}{1 - \frac{1}{\sqrt{3}} \cdot \frac{-1}{2}}$$

$$E. \tan\left(\frac{5\pi}{12}\right) = \tan(75^\circ) = \frac{\frac{\sqrt{3}}{3} + 1}{1 - \frac{\sqrt{3}}{3}} = \frac{\frac{\sqrt{3} + 3}{3}}{\frac{3 - \sqrt{3}}{3}} = \frac{\sqrt{3} + 3}{3} \cdot \frac{3}{3 - \sqrt{3}} = \frac{\boxed{\sqrt{3} + 3}}{\boxed{3 - \sqrt{3}}} = \frac{\boxed{3 + \sqrt{3}}}{\boxed{3 - \sqrt{3}}}$$

Example 4 Find the value of $\cos(x-y)$ if $0 < x < \frac{\pi}{2}$, $0 < y < \frac{\pi}{2}$, $\sin x = \frac{4}{9}$, $\sin y = \frac{1}{4}$

$$\begin{aligned} \cos(x-y) &= \cos x \cos y + \sin x \sin y \\ &= \frac{\sqrt{65}}{9} \cdot \frac{\sqrt{15}}{4} + \frac{4}{9} \cdot \frac{1}{4} \\ &= \frac{\sqrt{975}}{36} + \frac{4}{36} = \boxed{\frac{\sqrt{975} + 4}{36}} \end{aligned}$$

$$\begin{aligned} &\text{Right triangle with legs } 4 \text{ and } 1, \text{ hypotenuse } 9. \\ &\text{Pythagorean theorem: } m^2 + 1^2 = 9^2 \\ &m^2 + 1 = 81 \\ &m^2 = 65 \\ &m = \sqrt{65} \end{aligned}$$

$$\begin{aligned} &\text{Right triangle with legs } 4 \text{ and } 1, \text{ hypotenuse } 9. \\ &\text{Pythagorean theorem: } m^2 + 1^2 = 4^2 \\ &m^2 + 1 = 16 \\ &m^2 = 15 \\ &m = \sqrt{15} \end{aligned}$$

Example 5 Find the value of $\sin(x+y)$ if $0 < x < \frac{\pi}{2}$, $0 < y < \frac{\pi}{2}$, $\sin x = \frac{4}{5}$, $\sin y = \frac{5}{13}$

$$\begin{aligned} \sin(x+y) &= \sin x \cos y + \cos x \sin y \\ &= \frac{4}{5} \cdot \frac{12}{13} + \frac{3}{5} \cdot \frac{5}{13} \\ &= \frac{48}{65} + \frac{15}{65} = \boxed{\frac{63}{65}} \end{aligned}$$