Shortest Path
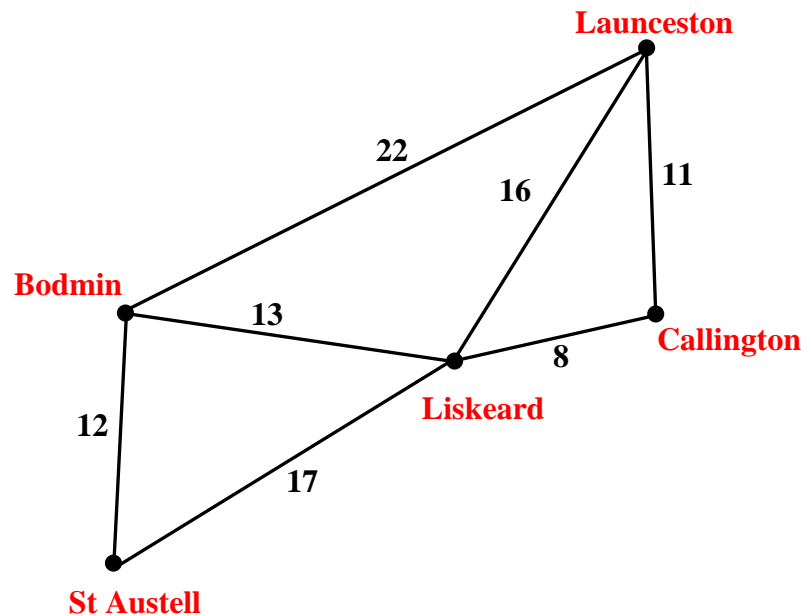
The network and matrix below give estimated distances between some places in Cornwall.

Distance matrix

	B	St	Li	C	La
B	0	12	13	0	22
St	12	0	17	0	0
Li	13	17	0	8	16
C	0	0	8	0	11
La	22	0	16	11	0

Suppose you want to find the shortest distance from St. Austell to Launceston.
In a simple network such as this, it is easy to see that the route via Liskeard will be the shortest.
But in more complex networks it is often much more difficult to find the shortest or quickest route. One method that will always work is called Dijkstra's Algorithm:

Dijkstra's Algorithm

Step 1 Label the starting vertex as 0 and *box* it to make it a *permanent* label.

Step 2 Write temporary labels showing the distance to each vertex that is connected directly to the starting vertex.

Step 3 Find the vertex with the smallest temporary label, and *box* it to make it *permanent*.

Step 4 From this vertex, find the total distance to each directly connected vertex and write as temporary labels. If a longer distance is already at a vertex, cross it out and replace it with the shorter distance. Then find the smallest temporary label and *box* it to make it *permanent*.

Step 5 Repeat Step 4 until you have given a permanent label to the finishing vertex.

Step 6 Use the permanent labels to trace back through the network to find the shortest path.

How Dijkstra's algorithm gives the solution to the example above is shown overleaf.
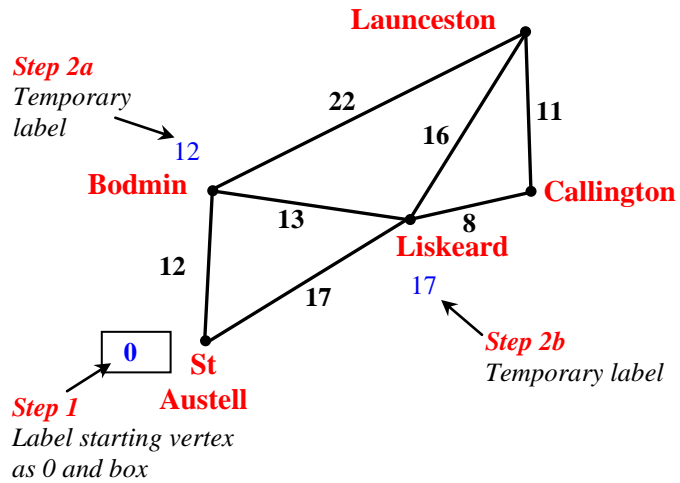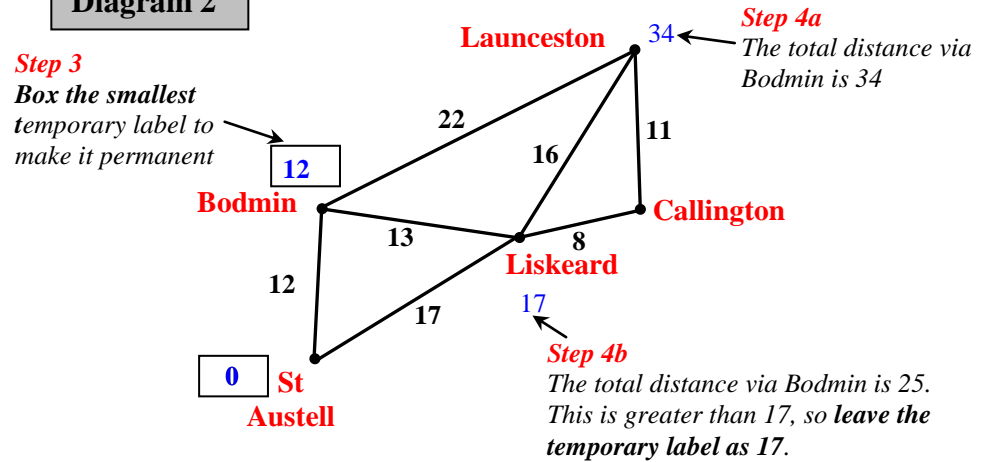
Diagram 1

Step 2a
Temporary label

Launceston

22   11

16

12

Bodmin

13   8

Callington

12   Liskeard

17   17

Step 2b
Temporary label

0

St Austell

Step 1
Label starting vertex as 0 and box

Diagram 2

Step 3
Box the smallest *temporary label to make it permanent*

Launceston   34

Step 4a
The total distance via Bodmin is 34

22   11

16

12

Bodmin

13   8

Callington

12   Liskeard

17   17

Step 4b
*The total distance via Bodmin is 25. This is greater than 17, so **leave the temporary label as 17**.*

0   St Austell

Diagram 3

Step 5a (repeat Step 4)
Distance via Liskeard, 33, is lower than the previous temporary label of 34 so replace it.

33
34

Launceston

22   11

16

12

Bodmin

13   8

Liskeard   Callington

12   25

17   *Step 5b (repeat Step 4)*
Add temporary label from Liskeard.

17

0   St Austell

Step 4c
Box the smallest *temporary label to make it permanent*

Diagram 4

33
34

Step 5d (repeat Step 4)
In this case this is the final box that is made permanent.

Launceston

12   22   11

Bodmin   16

13   8   Callington

12   Liskeard   25

17   17   *Step 5c (repeat Step 4)*
Box the smallest *temporary label to make it permanent.*

0   St Austell

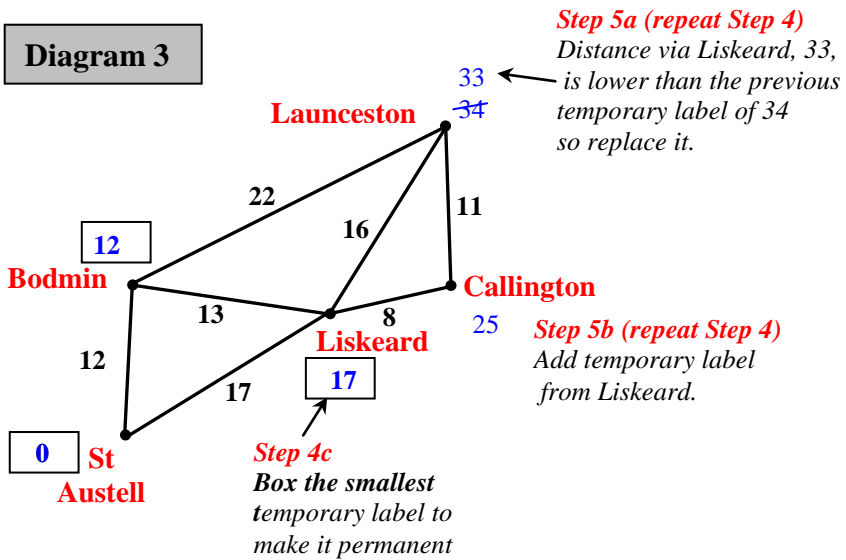**The shortest route from St Austell to Launceston is via Liskeard. The total distance is 33 miles.**

Photo-copiable
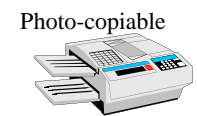
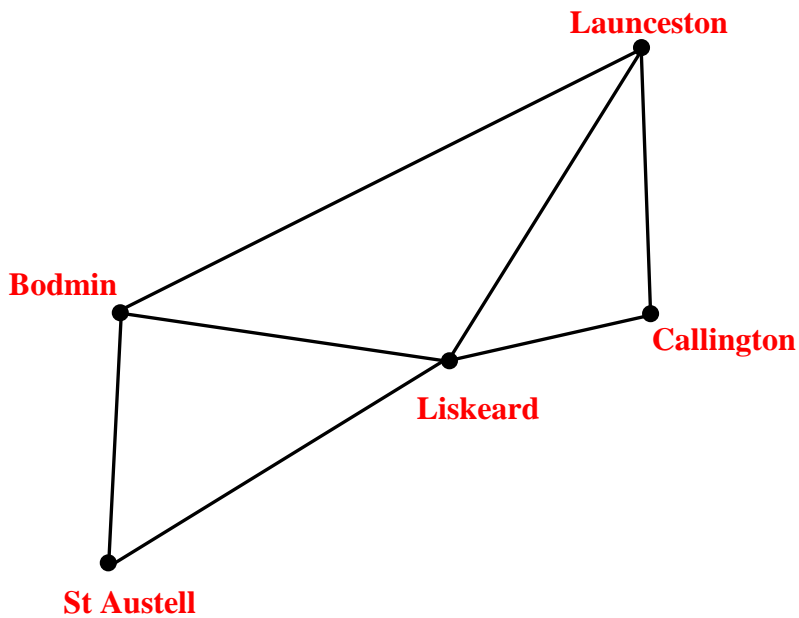# Shortest Path

Quickest Route

The road network for the places in Cornwall is shown again below.
In this case the matrix gives estimates of the travelling **times** in minutes between them.

Suppose you wish to use the **quickest** route from St. Austell to Launceston (rather than the shortest route). Use Dijkstra's algorithm to find the quickest route.

Launceston

Bodmin

Callington

Liskeard

St Austell

Time matrix

|	B	St	Li	C	La
B	0	15	16	0	22
St	15	0	21	0	0
Li	16	21	0	10	32
C	0	0	10	0	14
La	22	0	32	14	0

Shortest Path

Quickest Route

A more complex road network is given below.
The vertices represent places in the North West of England (see key below).
The weights give the estimated times in minutes that it takes to travel between these places.

Key			
A - Arclid	AE - Alderley Edge	B - Bosley	BC - Broken Cross
Ch - Chelford	C - Congleton	D - Davenham	HC - Holmes Chapel
J - Junction	K - Knutsford	Mc - Macclesfield	M - Middlewich
MH – Monk's Heath	S - Sandbach	T - Tabley	Wl - Wilmslow W - Winsford

Use Dijkstra's algorithm to find the quickest route from Winsford (W) to Macclesfield (Mc).
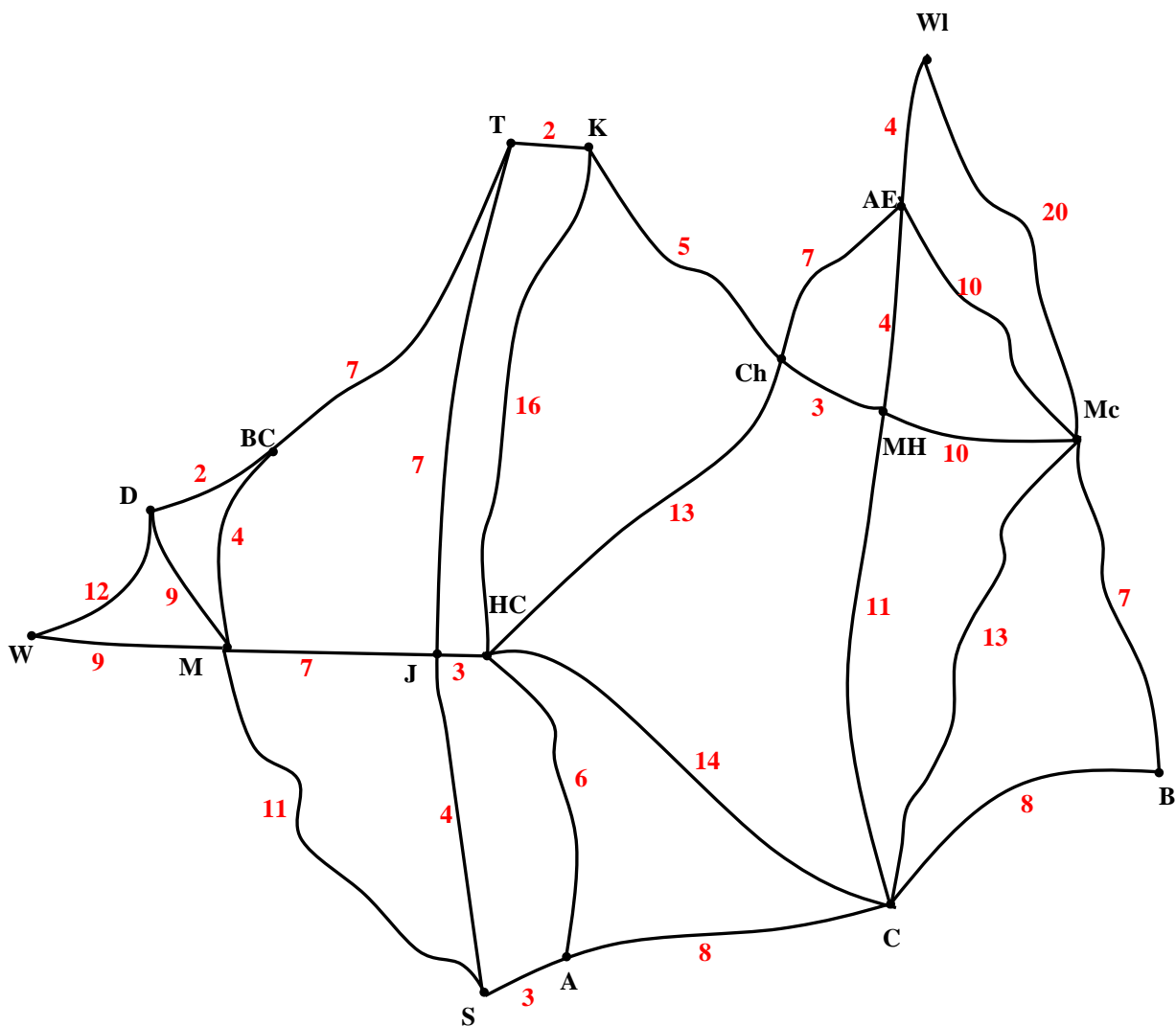
Shortest Path

Shortest Route

The North West road network is shown again below with weights on the arcs giving distances. Use Dijkstra's algorithm to find the shortest route from Winsford (W) to Macclesfield (Mc).

Key

|			
A - Arclid	AE - Alderley Edge	B - Bosley	BC - Broken Cross
Ch - Chelford	C - Congleton	D - Davenham	HC - Holmes Chapel
J - Junction	K - Knutsford	Mc - Macclesfield	M - Middlewich
MH – Monk's Heath	S - Sandbach	T - Tabley	Wl - Wilmslow W - Winsford
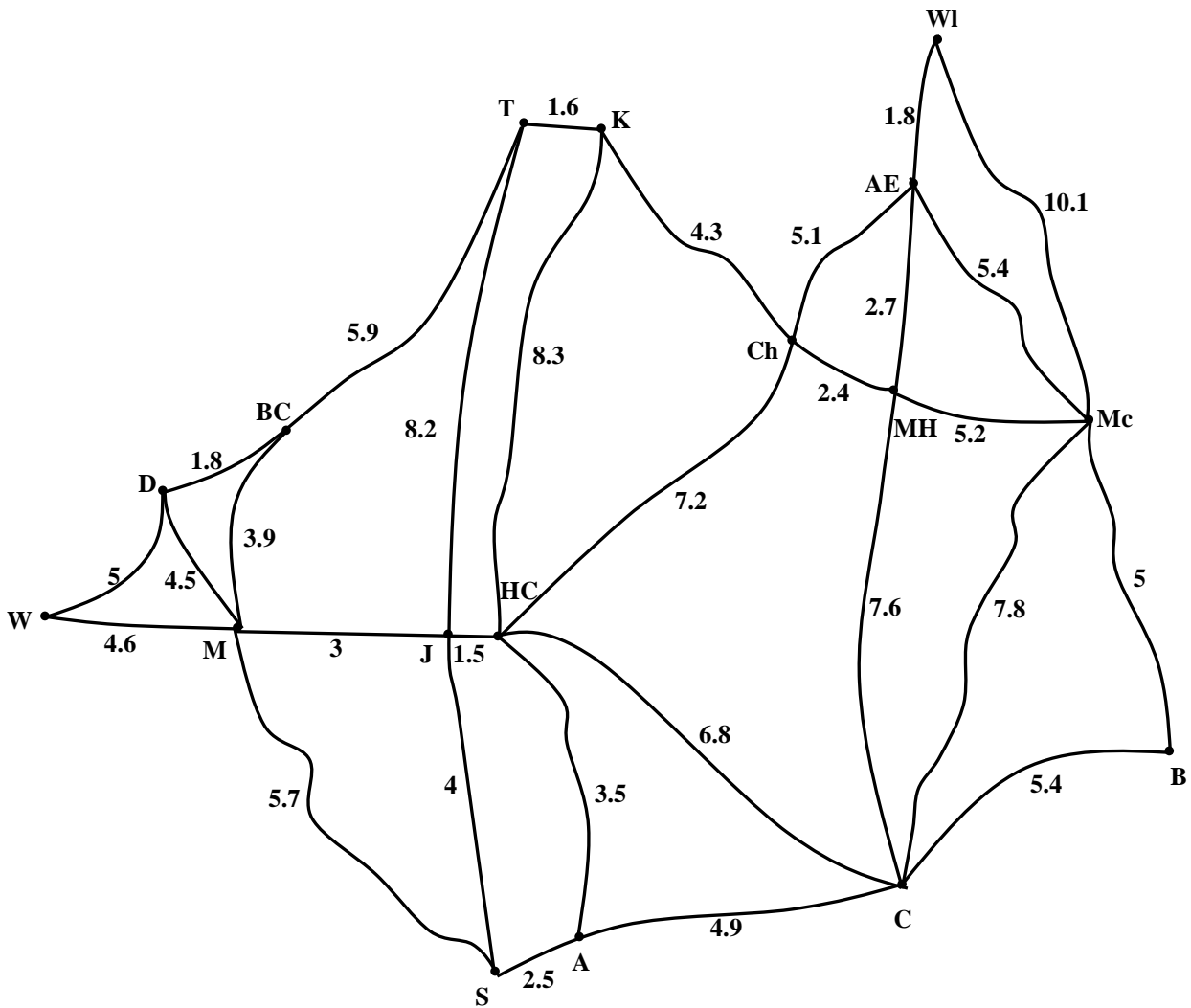
Shortest Path

Cycling Race

This network shows roads that cyclists can use in a cycle race.

All cyclists must start at A and ride to L as quickly as possible. They can use any route they wish.

Estimated times in minutes for each section of road are given on the diagram.
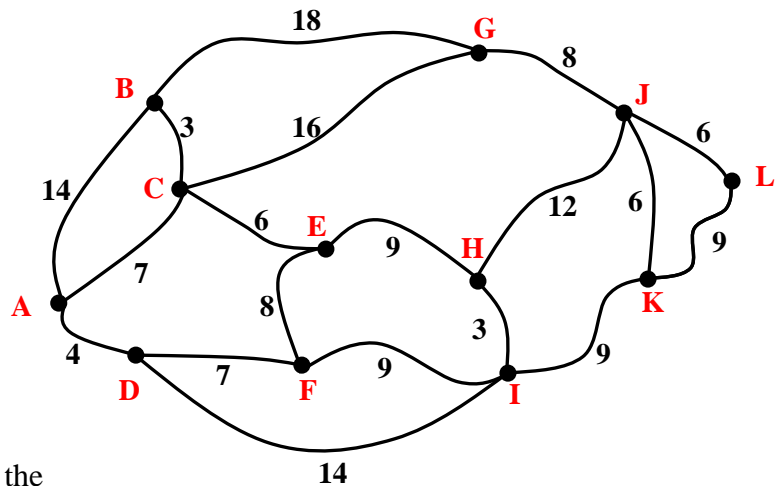
a) Use Dijkstra's algorithm to find the quickest route and estimate the total time.

b) Suggest reasons why in reality this route may not be the quickest and the estimated total time may not be very accurate.

Cheapest Airfare

The network below gives some of the air routes that can be travelled between the UK and New Zealand. The weights on the arcs show return fares in UK pounds.
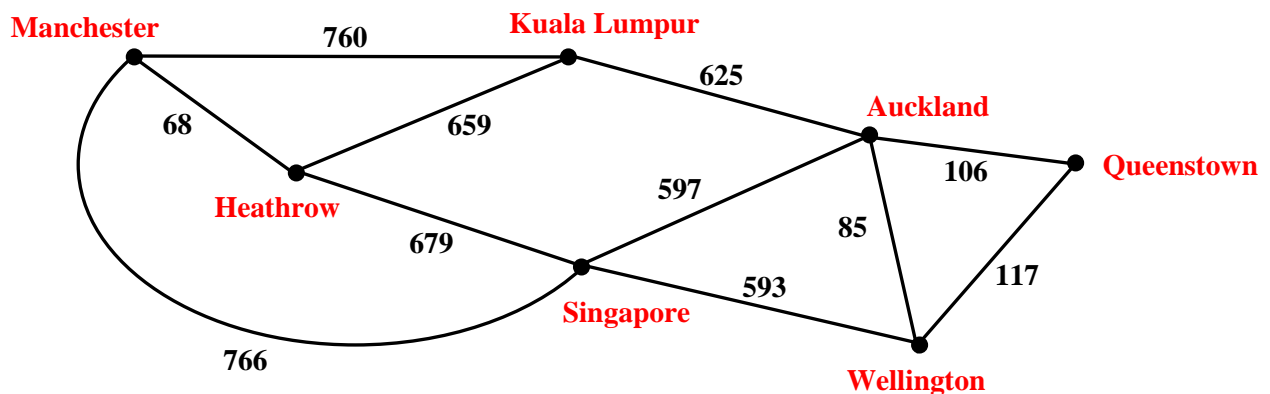
a) Use Dijkstra's algorithm to find the cheapest route from Manchester in the UK to Queenstown in New Zealand.

b) What other considerations might affect the decision on which route to take?

Unit   Advanced Level, Using and applying decision mathematics

Notes on Activity
This activity introduces Dijkstra's algorithm for finding the shortest path in a network. The Powerpoint presentation with the same name can be used to introduce the examples on pages 1 to 4 or to check learners' answers if they attempt the problems first.
The first Powerpoint slide shows how information from a matrix can be transferred to a network diagram and the second slide outlines Dijkstra's algorithm before it is used on slide 3 to show the shortest distance solution. You could then ask learners to solve the shortest time problem introduced on slide 4 using the worksheet on page 3 of this document before showing them the solution on slide 5. The problem that follows on slide 6 involves a much more complex network. It is suggested that this is demonstrated in stages. On showing the first temporary label introduce the idea of including the previous vertex – this is recommended by the Principal Examiner to make tracing the path much easier at the end of the problem. Learners could use page 4 of this document to show what they would do at each stage before you demonstrate it. You will need to discuss with learners what they should do if there are two temporary labels that have the same shortest distance (as occurs at T and S). The quickest time problem using the same network is introduced on slide 7 and learners can work through this on the worksheet on page 5.
The last two examples given on the worksheet on page 6 give extra practice using other contexts.

Answers
Pages 1 & 2
Shortest Route from St. Austell to Launceston is via Liskeard, a total of 33 miles.
Page 3
Quickest Route from St. Austell to Launceston is via Bodmin taking in total 37 minutes.
Page 4
Quickest Route from Winsford to Macclesfield takes 40 minutes
         W – M – BC – T – K – Ch – MH – Mc
Page 5
Shortest Route from Winsford to Macclesfield is 23.7 miles long
         W – M – J – HC – C – Mc
Page 6
Cycling Race
a)   Quickest Route: A – D – I – K – L has an estimated time of 36 minutes.
b)   The estimated times may not be accurate because of different road conditions, cyclists' fitness, wind and weather, how many cyclists are using the same route etc.
Cheapest Air Fare
a)   Cheapest Route is: Manchester – Heathrow – Singapore – Auckland – Queenstown. The total cost by this route is £1450.
b)   There are many other considerations that may affect the decision and these will depend on the purpose of the journey. Possible suggestions include flight times, the number of stops, the overall time taken, the possibility of breaking the journey to spend some time at the places visited (for holiday or business purposes) etc.

Note
Many of the estimated distances and times in this activity were found using the AA Route Planner at www.theaa.com/travelwatch/planner_main.jsp. This gives maps as well as distances and times. These could be used by your students in an assignment involving your local road network.