Derivatives of Inverse

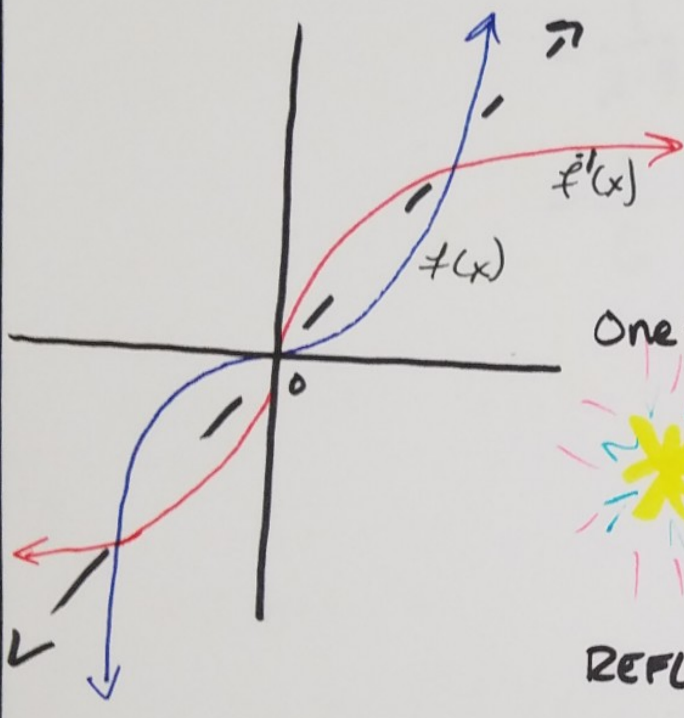
Quick Recap

FIND: 1. SWITCH x & y

2. SOLVE FOR y.

VERIFY: COMPOSITION OF FUNCTIONS

$f(f^{-1}(x)) = x$
$f^{-1}(f(x)) = x$ } ARE INVERSES



One-to-One: f PASSES VLT AND HLT

f's DOMAIN is f^{-1} RANGE

f's RANGE is f^{-1} DOMAIN

REFLECT OVER $y = x$. [see Diagram]

EXAMPLE: $f(x) = x^2$ (Does not pass HLT, MUST RESTRICT domain)

$x \geq 0$

FIND SLOPE OF TANGENT AT $x=3$

↳ I.R.O.C ⇒ 1ST DERIVATIVE.

$f'(x) = 2x$
$f'(x) = 2(3) = 6$

$f^{-1}(x) = \sqrt{x}$
$f^{-1}(x) = x^{\frac{1}{2}}$

① $y = x^2$
② $x = y^2$
③ $\sqrt{x} = y$

why not + or − HERE?

$$(f^{-1})'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{9}}$$

$$= \frac{1}{2 \cdot 3}$$

$$= \frac{1}{6}$$

FIND TANGENT at $x=3$

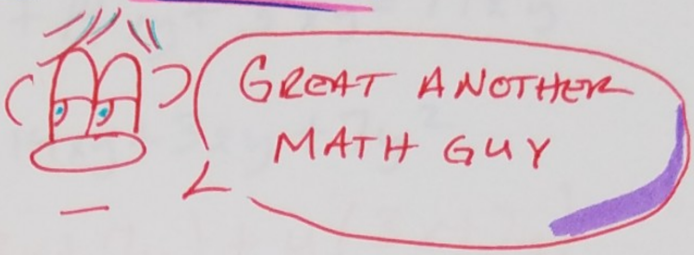
$y = x^2$

$9 = 3^2$

$f^{-1}(y)$ $f(x)$

Derivative of Inverse Function

LEIBNIZ



$$D_x\left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))}$$

y-value of f^{-1}

Range of function

EXAMPLE: $f(x) = x^2$ $[3, 9]$

$f'(x) = 2x$

$$D_x\left[f^{-1}(x)\right] = \frac{1}{2x} = \frac{1}{2\cdot 3} = \frac{1}{6}$$

EXAMPLE: $6x^2 + 17xy + 7y^2$

$1 _ \quad _ \quad _ \quad _ : $

42

$+14xy + 3xy = 17xy$

$6x^2 + 14xy + 3xy + 7y^2$

$2x(3x+7y) + y(3x+7y)$

$(2x+y)(3x+7y)$

$\left(2+\frac{dy}{dx}\right)(3x+7y) + \left(3+7\frac{dy}{dx}\right)(2x+y)$

$6x + 14y + 3x\frac{dy}{dx} + 7y\frac{dy}{dx} + 6x + 3y + 14\frac{dy}{dx} + 7y\frac{dy}{dx}$

$\left(3x\frac{dy}{dx} + 7y\frac{dy}{dx} + 14\frac{dy}{dx} + 7y\frac{dy}{dx}\right) + (6x + 14y + 6x + 3y)$

$\left(3x\frac{dy}{dx} + 14y\frac{dy}{dx} + 14\frac{dy}{dx}\right) + (12x + 17y)$

$\frac{dy}{dx}\left[3x + 14y + 14\right] + (12x + 17y)$

Hannah's Fun Problem of Unicorns

EXAMPLE: $f(x) = x^3 + 2x - 10$

① Switch x & y

$$x = y^3 + 2y - 10$$

② TAKE DERIVATIVE [IMPLICIT DIFFERENTIATION]

$$1 = 3y^2\left(\frac{dy}{dx}\right) + 2\left(\frac{dy}{dx}\right) - 0$$

$$1 = \frac{dy}{dx}\left(3y^2 + 2\right)$$

$$\frac{1}{3y^2+2} = \frac{dy}{dx}$$

EXAMPLE: $y = 2x^5 + x^3 + 1 \quad f \to (1,4)$

$$f'\left(f^{-1}(x)\right) = \frac{dy}{dx} = \frac{1}{10y^4 + 3y^2} \text{ or } \frac{dx}{dy}$$

$$= \frac{1}{10(1) + 3(1)} = \frac{1}{13}$$

$f(x) = 2x^5 + x^3 + 1$ $\qquad f \to (1,4)$

$f(1) = 2(1)^5 + (1)^3 + 1 = 4$

$f'(1) = 10(1)^4 + 3(1)^2 = 13$

$\dfrac{dy}{dx} = 13$

Recip.

$\dfrac{dx}{dy} = \dfrac{1}{13}$

The IROC of the inverse at a given point equals the reciprocal of the IROC of the original function at the same exact point.

$f \to (1,4)$

$f^{-1} \to (4,1)$

Reciprocal of Derivatives

EXAMPLE 4 on packet

IF $f(2)=-3$ $\quad f'(2)=\frac{3}{4}$

$(2,-3)$

$g(x)=f^{-1}(x)$

equation of tangent line.

I.R.O.C f^{-1} @ $(-3,2)$ is $\frac{4}{3}$

$x \quad y \qquad m$

$y-y_1=m(x-x_1)$

$y-2=\frac{4}{3}(x+3)$

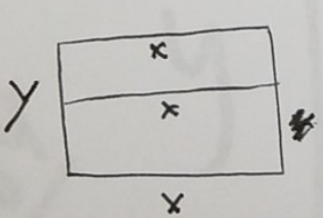
Optimization Problems

2. 384 meter² plot

$A=l \times w = 384\,m^2$

$l \to y, \quad w \to x$

$y=\frac{384}{x}$



$P=3x+2y$

$P=3x+2\left(\frac{384}{x}\right)$

$P=3x+\frac{768}{x}$

$P'=3+\frac{-768}{x^2}$

$P'=\frac{3x^2-768}{x^2}$

$0=3x^2-768$

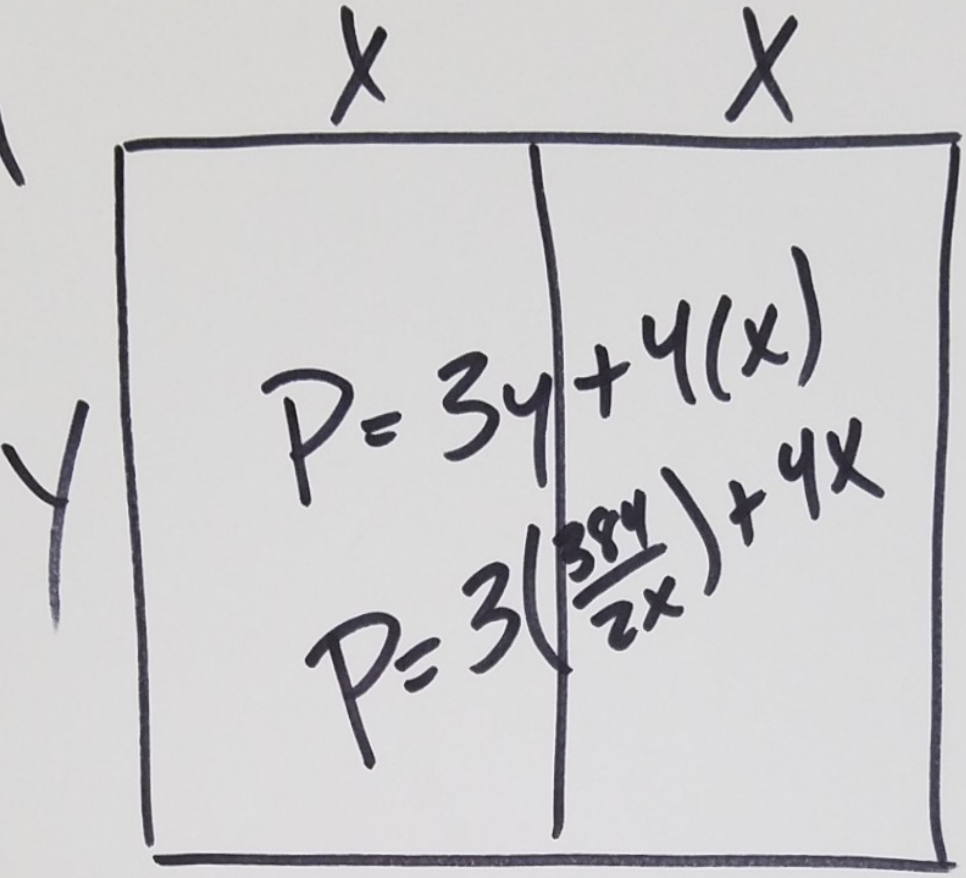
$768=3x^2$

$256=x^2$

$x=\sqrt{256}$

$x=16$

$y=24$

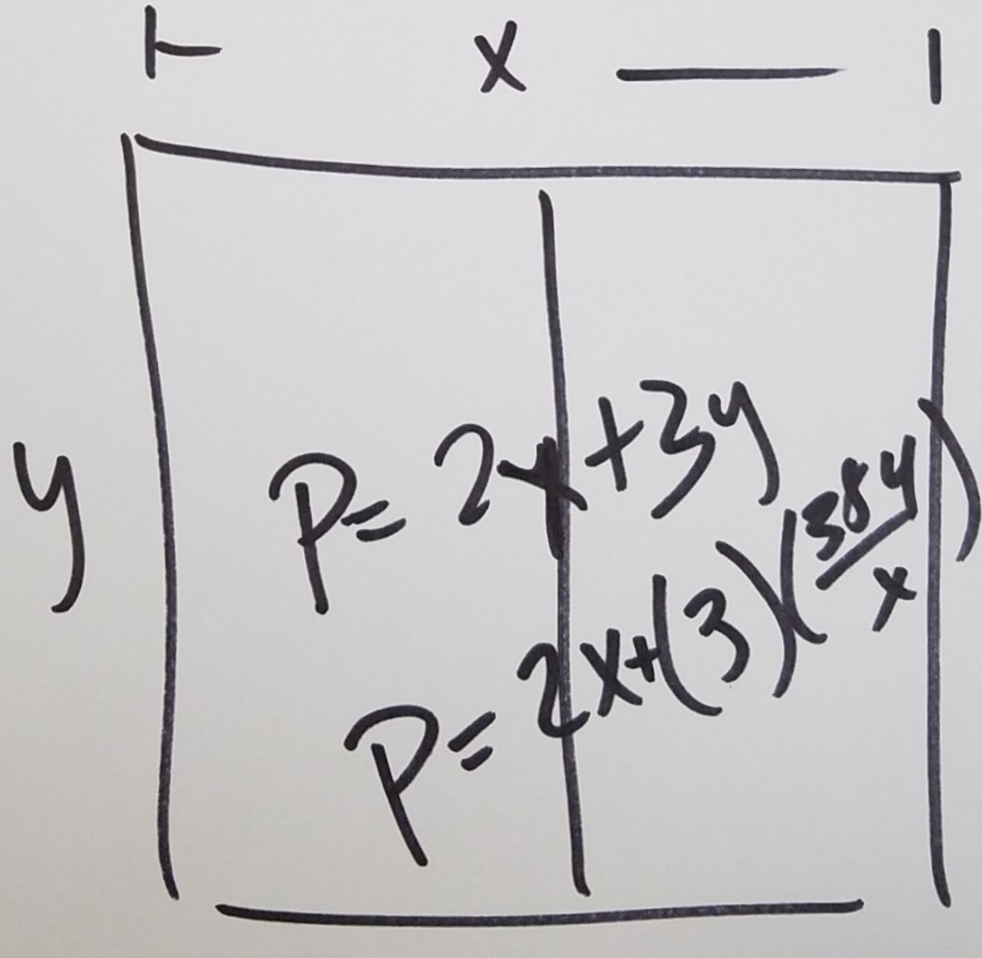


$A = 2x \cdot y$

$\frac{384}{2x}$

$P = 3y + 4(x)$

$P = 3\left(\frac{384}{2x}\right) + 4x$



$A = x \cdot y$

$384/x$

$P = 2x + 3y$

$P = 2x + (3)\left(\frac{384}{x}\right)$