Warm Up

- Use taking turns on the following with Ivana choosing first:

Donald	**Ivana**
Palm Beach Mansion	Connecticut Estate
Trump Tower Triplex	Palm Beach Mansion
Connecticut Estate	Trump Plaza Apartment
Trump Plaza apartment	Trump Tower Triplex
Cash and Jewelry	Cash and Jewelry

Warm Up

Prince Michael, Paris, and Blanket are dividing up some of their dad's assets. They decided to use Knaster Inheritance to perform a fair division. Find what each child gets and walks away with.

	White Glove	Beatles Copyrights
Prince Michael	$3,000	$15,000
Paris	$5,100	$14,100
Blanket	$6,000	$12,000

Answer

Prince Michael: Beatles copyright – $8133.34

Paris: $7266.66

Blanket: Glove + $866.66

Fair Division

Lesson 3

Cake Cutting Procedures

Cake Cutting Procedures: Proportionality

- Definition: a cake cutting procedure for *n* players can be used to allocate a cake among themselves (no outside arbitrators) so that each player has a strategy that will guarantee that player a piece with which he or she is "satisfied" with even in the face of collusion by the others.

Cake Cutting Procedures: Proportionality

- A cake cutting procedure (for *n* players) is proportional if each player's strategy guarantees that a player a piece of size or value at least of 1/*n* of the whole in his or her own estimation. It will be called envy-free, as in the adjusted winner context, each player's strategy guarantees that player a piece he or she considers to be at least tied for largest or most valuable.

Cake Cutting Procedures: Proportionality

- Many procedures that are proportional fail to be envy-free
- Thus proportional procedures are fairly easy to come by, but envy-free procedures are fairly hard to come by

Divide and Choose

- I cut, you choose.
- Can be done between 2 people.

Lone Divider

- AKA: The Steinhaus Proportional Procedure for Three Players
- We have three players – Bob, Carol, and Ted
- Bob cuts the cake into 3 pieces – *X*, *Y*, and *Z*

Lone Divider

- Carol is going to approve a piece of she believes to be at least 1/3 in size or value
- Ted will do the same
- If there are distinct pieces, say *X* and *Y*, with Carol approving of *X* and Ted approving of *Y*, then we give the third piece, *Z*, to Bob.
- In this situation, we would be done

Lone Divider

- This time, let's assume Carol and Ted only approve of one piece, *X*
- And (which is more important to us), both disapprove of *Z*
- Let *XY* denote the result of putting piece *X* and *Y* back together to form a single piece
- Thus, we can give *Z* to Bob and let Carol and Ted use divide-and-choose on *XY*.
- Because half of 2/3 is 1/3, both Carol and Ted are guaranteed a proportional share (as is Bob, who approved all three pieces)

Last Diminisher

- AKA: The Banach-Knaster Proportional Procedure for Four or More Players
- Bob cuts from the cake a piece that he thinks is of size one-fourth and hands it to Carol
- If Carol thinks the piece is larger than ¼ she trims it and adds the trimmings back to the cake

Last Diminisher

- If Carol thinks the piece handed to her is of size at most ¼, she passes it unaltered to Ted
- Ted proceeds as Carol did and then passes to Alice
- Alice proceeds the same as well, but waits a moment before passing it on
- If no one has altered the piece, it will go back to Bob. If someone has altered the piece, it will go to the last person who altered it.

Last Diminisher

- So, if Ted were the last person to trim the piece, he would retain the piece
- We would repeat the procedure with the remaining (at least) ¾ of cake left.
- Then, when ½ of the cake is left, the remaining two players would revert to divide-and-choose

Cake-Division Procedures: The Problem with Envy

- Divide and choose has a property that neither one of the last procedures possesses: It can ensure each player receives a piece that he or she considers the largest or at least tied for the largest. This makes divide and choose envy-free.

Selfridge-Conway Envy-Free Procedure

- We start with a cake and three people
- We want to arrive at an envy-free allocation of the entire cake among three people in a finite number of steps.

Selfridge-Conway Envy-Free Procedure

- Steps
  1. Player 1 cuts the cake into three pieces he considers to be the same size. He hands the three pieces to player 2.
  2. Player 2 trims at most one of the three pieces to create at least a two-way tie for largest. Setting the trimmings aside, player 2 hands the three pieces (one of which may have been trimmed) to player 3.

Selfridge-Conway Envy-Free Procedure

3. Player 3 now chooses, from among the pieces, one that he considers to be at least tied for largest.
4. Player 2 next chooses, from the two remaining pieces, one that she considers to be at least tied for largest, with the proviso that if she trimmed a piece in step 2, and player 3 did not choose this piece, then she most now choose it.
5. Player 1 receives the remaining piece.

Questions about Cake Cutting

1. Two people use the divide and choose procedure to divide a field. Suppose Jeff divides and Karen chooses. Which statement is true?
    1. Karen believes she gets more than her fair share.
    2. Karen can guarantee she always gets at least her fair share.
    3. Karen can possibly believe she gets less than her fair share.
2. Suppose 7 people will share a cake using the last diminisher method. To begin, Scott cuts a piece and passes it to the 6 other people, but no one trims the piece, what happens next?
    1. Scott gets the piece.
    2. The last person who is handed the piece keeps it.
    3. The piece is returned to the cake and someone else cuts a piece.

Questions about Cake Cutting

3. Using Lone Divider (3 people), what happens if there is a single portion that is the only one approved by both non dividers?
    1. One of the other portions is given to the divider.
    2. The two non dividers flip a coin to see who receives the approved portion.
    3. All portions are returned to the cake and a new divider is chosen.
4. Using Lone Divider, what happens if the 2 non dividers choose different portions?
    1. Each non divider chooses a portion he or she has approved.
    2. The divider receives his or her choice of the portions.
    3. The divider selects portions for each person.

Questions about Cake Cutting

5. Using Last Diminisher (4 or more) what happens to the first portion after everyone has inspected it and possible trimmed it?
    1. The portion goes to the last person to approve, whether or not it was trimmed.
    2. The portion goes to the last person to trim.
    3. The portion goes to the first person to approve and not trim the portion.
6. Using Last Diminisher, what does the player who receives the first portion then do?
    1. That player becomes the first to approve the 2nd portion.
    2. That player becomes the last person to approve the 2nd portion.
    3. That person leaves the game.

Questions about Cake Cutting

7. Using Last Diminisher, what happens to the last 2 people who remain?
    1. One player chooses the scraps, the other player chooses the portion.
    2. One player separates the remainders into 2 portions and the other person chooses.
    3. The players flip a coin to decide who takes the remainders.
8. Using the Selfridge-Conway Envy Free procedure for 3 players, which of the following statements is true?
    1. Each player has the opportunity to trim the portions if they appear to be unfair.
    2. Each player receives a portion that he or she believes to be exactly 1/3 of the total.
    3. The first player may believe the third player received more than a fair share.

Questions about Cake Cutting

9. For Selfridge Conway, which of the following statements is true?
    1. The first player will receive one of the 3 portions originally cut.
    2. The second player will always receive one of the trimmed portions.
    3. The third player will always believe he has received exactly his fair share.
10. A fair division procedure is “envy-free” when each player believes that
    1. Each player has received an equal portion.
    2. He or she received at least a fair share.
    3. No other player received more than he or she did.