Power Series (centered at c)

Last unit → talked about series of constants
This unit → talking about series of functions

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c)^1 + a_2(x-c)^2 + \dots$$

if $c = 0$: $a_0 + a_1 x + a_2 x + a_3 x \dots$

▸ convergence of a power series centered at "c" ◂

~ only ONE of the following occurs ~

① series converges only at c — R.O.C = 0

② series converges absolutely for $\forall x$ — R.O.C = ∞

③ series converges for an interval of values — R.O.C = $\frac{1}{2}$(length of I.O.C.)

R.O.C. : Radius of convergence
I.O.C. : interval of convergence

↳ the R.O.C. is the distance from the midpoint of the interval of convergence to the end point.

Ex ① find I.O.C and R.O.C.

$$\sum_{n=0}^{\infty} n \cdot \left(\frac{2x-5}{3}\right)^n = 0 + \left(\frac{2x-5}{3}\right)^1 + 2\left(\frac{2x-5}{3}\right)^2 + \dots$$

Next ... ratio test!

$$\lim_{n\to\infty}\left|\frac{(n+1)\left(\frac{2x-5}{3}\right)^{n+1}}{n\left(\frac{2x-5}{3}\right)^{n}}\right| =$$

$$\lim_{n\to\infty}\left|\frac{(n+1)}{(n)}\cdot\left(\frac{2x-5}{3}\right)^{1}\right| =$$

$$1\cdot\left|\frac{2x-5}{3}\right| = \left|\frac{2x-5}{3}\right| \Rightarrow \text{it converges if } \left|\frac{2x-5}{3}\right| < 1$$

[Do you remember solving absolute value equations]

case 1:

$$\frac{2x-5}{3} < 1$$

case 2:

$$\frac{2x-5}{3} > -1$$

so...

$$-1 < \frac{2x-5}{3} < 1$$ ① multiply by 3

$$-3 < 2x-5 < 3$$ ② add 5

$$2 < 2x < 8$$ ③ divide by 2

$$1 < x < 4$$

Now check your endpoints.

when $x=1$ $\quad \frac{2(1)-5}{3} = -\frac{3}{3} = -1$

when $x=4$ $\quad \frac{2(4)-5}{3} = \frac{3}{3} = 1$

$x=1 \quad \sum_{n=0}^{\infty} n(-1)^n$ alternating series

$\lim_{n\to\infty} n = \infty$ diverges by n^{th} term test

$x=4 \quad \sum_{n=0}^{\infty} n(1)^n = \sum_{n=0}^{\infty} n$

$\lim_{n\to\infty} n = \infty$ diverges by n^{th} term test

$$\boxed{\begin{array}{l} \text{I.O.C.} = (1,4) \\ \text{R.O.C.} = \frac{1}{2}(3) = 1.5 \end{array}}$$

Ex ② $\sum_{n=0}^{\infty} (2n)! \left(\frac{x}{2}\right)^n = 1 + 2\left(\frac{x}{2}\right) + \underbrace{24\left(\frac{x}{2}\right)^2}_{\omega x^2} + \dots$

$$\lim_{n\to\infty} \left| \frac{(2(n+1))! \left(\frac{x}{2}\right)^{n+1}}{2(n)! \left(\frac{x}{2}\right)^n} \right|$$

¡Remember!

$(2(n+1))! = (2n+2)! = (2n+2)\ (2n+1)(\ 2n\ ) \dots$

$$\lim_{n\to\infty} \left| \underline{(2n+2)(2n+1)} \left(\frac{x}{2}\right)^1 \right| \longrightarrow \text{after simplification}$$

$= \infty \cdot \left|\frac{x}{2}\right| = \infty$ Diverges

which means $\left|\frac{x}{2}\right| . \infty$ if $x = 0$ THEN it converges

$$\boxed{\begin{array}{l} \text{I.O.C.} = 0 \\ \text{R.O.C.} = \frac{1}{2}(0) = 0 \end{array}}$$

Here's how you simplify:

$(2n+2)! = (2n+2)(2n+1)(\cancel{2n})(\cancel{2n-1})(\cancel{2n-2}) \dots$

$(2n)! = (\cancel{2n})(\cancel{2n-1})(\cancel{2n-2}) \dots$

After dividing to one, you are left with

$\underline{(2n+2)(2n+1)}$.

Ex ③ $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots$

$$\lim_{n\to\infty} \left| \frac{\dfrac{x^{2(n+1)+1}}{(2(n+1)+1)!}}{\dfrac{x^{2n+1}}{(2n+1)!}} \right| = \lim_{n\to\infty} \left| \frac{x^{2n+3}}{x^{2n+1}} \cdot \frac{(2n+1)!}{(2n+3)!} \right|$$

$$\lim_{n\to\infty} \left| x^2 \cdot \frac{1}{(2n+3)(2n+2)} \right| = x^2 \cdot 0 = 0$$

0 is always < 1 so converges for $\forall x$

I.O.C. = $(-\infty, \infty)$

R.O.C. = (∞)

Ex ④ $\displaystyle\underbrace{\sum_{n=0}^{\infty} \frac{x^n}{n!}}_{\text{power series for } e^x}$, find $f'(x)$

$$f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots \qquad + e^x$$

$$f'(x) = 0 + 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots \qquad + e^x$$

$$f'(x) = \sum_{n=}^{\infty} \frac{n \cdot x^{n-1}}{n!} \quad \text{or} \quad \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$

Ex ④ $\displaystyle\sum_{n=}^{\infty} \frac{x^n}{n^2} = x + \frac{x^2}{4} + \frac{x^3}{9} + \dots$

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)^2}}{\frac{x^n}{n^2}} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \cdot \frac{n^2}{(n+1)^2} \right|$$

$$\lim_{n\to\infty} \left| x^1 \cdot \left(\frac{n}{n+1}\right)^2 \right| = |x| \cdot 1$$

converges if $|x| < 1$

$-1 < x < 1$

Check endpoints:

$x = 1$ $\displaystyle\sum_{n=1}^{\infty} \frac{(1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series

$x = -1$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \sum_{n=1}^{\infty} (-1)^n a_n$ converges by alt. series

$$\boxed{\begin{array}{l} \text{I.O.C. } [-1, 1] \\ \text{R.O.C. } = 1 \end{array}}$$