

## Taylor Series

If a function  $f(x)$  is infinitely differentiable at  $c$ , the power series centered at  $c$  is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$$

$$= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \dots$$

- if  $c = 0$ , it is a MacLaurin Series

Ex. ①  $f(x) = e^{3x}$  Find the MacLaurin series  
↳ when  $c = 0$

$$f(x) = e^{3x} = e^{3(0)} = 1$$

$$f'(x) = 3e^{3x} = 3e^{3(0)} = 3$$

$$f''(x) = 9e^{3x} = 9e^{3(0)} = 9$$

$$f'''(x) = 27e^{3x} = 27e^{3(0)} = 27$$

$$f^{(4)}(x) = 81e^{3x} = 81e^{3(0)} = 81$$

$$1 + 3(x-0) + \frac{9}{2!}(x-0)^2 + \frac{27}{3!}(x-0)^3 + \frac{81}{4!}(x-0)^4 + \dots$$

- Time for some pattern recognition

$$1^{\text{st}}: 1 + 3 + 9 + 27 + 81 + \dots$$

$$3^0 + 3^1 + 3^2 + 3^3 + 3^4 + \dots \boxed{3^n}$$

$$2^{\text{nd}}: x^0 + x^1 + x^2 + x^3 + x^4 + \dots \boxed{x^n}$$

$$3^{\text{rd}}: 0! + 1! + 2! + 3! + 4! + \dots \boxed{n!}$$

$$\sum_{n=0}^{\infty} \frac{3^n x^n}{n!} \quad \text{-OR-} \quad \sum_{n=0}^{\infty} \frac{(3x)^n}{n!}$$

$$\text{so, if } e^{3x} = \sum_{n=0}^{\infty} \frac{3^n x^n}{n!}, \text{ then}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{4x} = \sum_{n=0}^{\infty} \frac{4^n x^n}{n!}$$

$$e^{7x} = \sum_{n=0}^{\infty} \frac{7^n x^n}{n!}$$

... etc.

Ex ②  $f(x) = \cos\sqrt{x}$  find maclaurin series

► first, some elementary functions :

$$1. e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{IOC: } (-\infty, \infty)$$

$$2. \ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n} \quad (0, 2]$$

$$3. \frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n \quad (0, 2)$$

$$4. \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \quad (-1, 1)$$

$$5. \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad (-\infty, \infty)$$

$$6. \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad (-\infty, \infty)$$

$$7. \arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \quad [-1, 1]$$

$$8. \arcsin x = \sum_{n=0}^{\infty} \frac{(2n)! x^{2n+1}}{(2^n n!)^2 (2n+1)} \quad [-1, 1]$$

so, if  $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ , then

$$\cos \sqrt{x} = \sum_{n=0}^{\infty} \frac{(-1)^n (\sqrt{x})^{2n}}{(2n)!} \quad \left. \begin{array}{l} (\sqrt{x})^{2n} = (x^{\frac{1}{2}})^{2n} \\ = x^n \end{array} \right\}$$

$$\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n (x)^n}{(2n)!}}$$



Ex. ③  $f(x) = \sin x$  find taylor series  $c = \frac{\pi}{4}$

$$f(x) = \sin x = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$f'(x) = \cos x = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$f''(x) = -\sin x = -\sin\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$f'''(x) = -\cos x = -\cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$f^{(4)}(x) = \sin x = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(x - \frac{\pi}{4}) + \frac{-\frac{1}{\sqrt{2}}}{2!}(x - \frac{\pi}{4})^2 + \frac{-\frac{1}{\sqrt{2}}}{3!}(x - \frac{\pi}{4})^3 + \dots$$

$$\text{GCF: } \frac{1}{\sqrt{2}}$$

$$\frac{1}{\sqrt{2}} \left( 1 + (x - \frac{\pi}{4}) - \frac{1}{2!} (x - \frac{\pi}{4})^2 - \frac{1}{3!} (x - \frac{\pi}{4})^3 + \dots \right)$$

Pattern Recognition:

signs  $\Rightarrow +0+1-2-3+4+5-6-7+8+9-10-11\dots$

$$(-1) \frac{n(n-1)}{2} \quad \begin{matrix} \uparrow & \text{EXPO} \\ \text{N} & \text{N-1} \end{matrix}$$

A) Ex. 3 cont.

$$\frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{(-1)^{\frac{n(n-1)}{2}}}{n!} \left(x - \frac{\pi}{4}\right)^n$$

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$$f(x) = e^{3x}$$

Also, "Maclaurin Series"

Ex. for  $f(x) = e^{3x}$  MacLaurin Series

$$f(0) = e^{0} \Rightarrow 1$$

$$f'(0) = 3e^{0} \Rightarrow 3$$

$$f''(0) = 9e^{0} \Rightarrow 9$$

$$f'''(0) = 27e^{0} \Rightarrow 27$$

$$f''''(0) = 81e^{0} \Rightarrow 81$$

$$1 + 3(x-0) + \frac{3}{2}(x-0)^2 + \frac{27}{3!}(x-0)^3 + \frac{81}{4!}(x-0)^4$$

Same for some Pattern Recognition